Kepler's third law of n-body periodic orbits in a Newtonian gravitation field

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This study considers the periodic orbital period of an n-body system from the perspective of dimension analysis. According to characteristics of the n-body system with point masses $(m_1, m_2, ..., m_n)$, the gravitational field parameter, $\alpha \sim Gm_i m_j$, the n-body system reduction mass M_n, and the area, A_n, of the periodic orbit are selected as the basic parameters, while the period, T_n, and the system energy, $|E_n|$, are expressed as the three basic parameters. By using Buckingham π-theorem of dimensional analysis, these two relations can be reduced to a dimensionless form, which can surprisingly produce only one dimensionless π, respectively. Because there is only one π, therefore the π must be a constant. Since the two-body system is a special case of the n-body, we can uniquely determine the two constants by using the two-body Kepler's third law. Thus, the n-body system Kepler's third law is deduced and is given by $T_n|E_n|^{3/2} = \frac{\pi}{\sqrt{2}} G \left(\frac{\sum_{i=1}^{n} \sum_{j=i+1}^{n} (m_i m_j)^3}{\sum_{k=1}^{n} m_k} \right)^{1/2}$. A numerical validation and comparison study was hence conducted.

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INTRODUCTION

One of the central and vivid problems of celestial mechanics in the 18th and 19th centuries was about the description of the motion of the Sun-Earth-Moon system under the Newtonian gravitation field, depicted in Figure 1. Notable work was done by Euler (1760), Lagrange (1776), Laplace (1799), Hamilton (1834), Liouville (1836), Jacobi (1843), and Poincaré (1889) [1] and Xia (1992) [2]. The study of the motion of two bodies was solved by Kepler (1609) and Newton (1687) early in the 17th century. For the elliptic periodic orbit of 2-body system in Fig 1, Kepler's third law of the two-body system [3] is given by $T|E|^{3/2} = \frac{\pi}{\sqrt{2}} G m_1 m_2 \sqrt{\frac{m_1 m_2}{m_1 + m_2}}$, where the gravitation constant, $G = 6.673 \times 10^{-11} m^3 kg^{-1} s^{-2}$, the orbit period, T, the total energy of the 2-body system, $|E|$, and point masses m_1 and m_2.

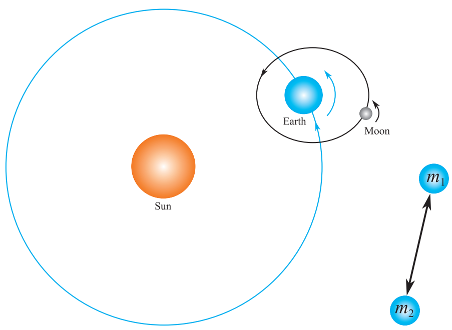


FIG. 1: The Sun-Earth-Mond system and 2-body system

However, the 3-body system depicted in Figure 2, cannot be solved analytically because unlike the 2-body problem, the 18 variables that describe the system cannot be reduced to a single variable. Simplification of the two-body problem was allowed by invariance and conserved quantities as 'first integrals'. It was proven that it is impossible to reduce the 18 variables of the 3-body problem in order to produce an analytic solution.

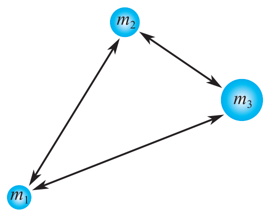


FIG. 2: 3-body system

Notwithstanding that the analytic solution cannot be found, it is possible to find a numerical solution for the 3-body problem, in which the study of the periodic 3-body orbit has received particular attention in recent years [4–11]. The figure-eight orbit was discovered numerically in 1993, using the principle of the least action by C. Moore [4]. Chenciner and Montgomery [5] rediscovered the solution by using the shape space least action principle in 2000. In 2013 Šuvakov and V. Dmitrašinović [6] made a breakthrough and produced epic results, they found 13 new distinct collisionless periodic orbits of the Newtonian planar 3-body system with an equal mass and zero angular momentum. In 2017 Li and Liao [10], Li, Jing and Liao [11] reported their breakthrough new finding: 695 periodic orbits of planar 3-body system with an equal mass and zero angular momentum, as well as 1223 periodic orbits of the planar 3-body problem with an unequal mass and zero angular momentum, respectively.

Corresponding to the new finding of more and more periodic obits, a fundamental law was discovered by nu-

merical experiments [9–11], which states that the 3-body system m_1, m_2 m_3) may obey a law, which is similar to the law of harmonies, name Kepler's third law. Analogous to the Kepler's third law of the two-body problem, Šuvakov and V. Dmitrašinović [9] proposed a generalized 3-body Kepler's third law: $T|E|^{3/2}$ = constant, where $|E|$ denotes the total kinetic and potential energy of the 3-body system, where T is the period of periodic orbit. However, they pointed out that "the constant on the right-hand-side of this equation is not "universal" in the 3-body case, as it is in the two-body case, and it may depend on both the family of the 3-body orbit and its angular momentum" [9]. Li and Liao [10] enhanced the aforementioned relation to $\bar{T}|E|^{3/2} = \bar{T}^*$, and numerically proved that $\bar{T}^*$ is approximately a universal constant, namely $\bar{T}^* = 2.433 \pm 0.075$, for the 3-body system with an equal mass and zero angular momentum. This remarkable scale-invariance period $\bar{T}^* = \bar{T}|E|^{3/2}$ was again proved by Li, Jing and Liao [11, 12] for the 3-body system with unequal mass and zero angular momentum, where $\bar{T}^* = 3.074m_3 - 0.617$ in the case of $m_1 = m_2 = 1$ and m_3 varied.

Although the aforementioned relations [9–11] were supported by the numerical experiments in Ref.[9–11], The questions still remain whether $T|E|^{3/2}$ = constant is an universal, and if not, what form it will takes and how to formulate it without a further numerical simulation. Would we be able to find similar relations or universal scaling laws, for a n-body system, and, if so, how?

The 2-body system incorporates Kepler's three laws, name the law of ellipses, the law of equal areas and the law of harmonies. Clearly, for the 3-body system, since the periodic orbit is no longer elliptical, so that there is no corresponding law of ellipses, and because the periodic orbital topology is more complex, the law of equal areas might also not established. From a large number of numerical simulations of the 3-body system, the time of each object walking along its orbit is the same, that is, for a given mass of the 3-body system, the periodicity of the periodic orbit in the gravitational field satisfies Kepler's third law, namely the law of harmonies.

The question now is whether or not the conclusions drawn from those limited numerical experiments are prevalent in the 3-body system. If so, can it possibly be extended to a n-body system depicted in Figure 3? Clearly, with an increase in the number of point mass, the system will have more and more degrees of freedom; while accordingly, the dynamic process becomes hugely complex. If you continue to use numerical simulation to study the n-body problem, the calculation will certainly become more and more challenge. Hence, we have no choice but to find an alternative approach.

This study has attempted to attack the n-body system (including 3-body) by using dimensional analysis [13–18]. The most powerful use of dimensional analysis is to predict the outcome of an numerical experiment, depending on the variables, whilst providing theoretical insight. Dimensional analysis may come across as simply trying to fit pieces of a puzzle together by trial and error. However, identifying the quantities that are relevant for a given problem is a demanding task, which requires deep physical insight [18]. This may be done as follows: make a list of all quantities on which the answer must depend, then write down the dimensions of these quantities, and finally demand that these quantities should be combined into a functional form that provides the right dimension. This scheme was cast into a formal framework by Buckingham in 1921 and is often referred to as the Buckingham π-theorem [19].

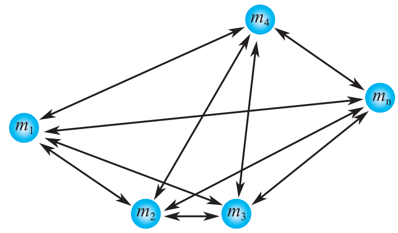


FIG. 3: n-body system

DIMENSIONAL FORMULATION

An important class of central gravitation fields is formed by those in which the potential energy is inversely proportional to the radius r, and the force accordingly inversely proportional to r^2. They include the fields of Newtonian gravitational and of Coulomb electrostatic interaction; the latter may be either attraction or repulsive. For the attractive gravitation field, the potential energy is $U(r) = -\alpha/r$, where α is a positive constant with dimension $[L]^3[M][T]$, which is from Newtonian 2-body potential [3].

Let's having a n-body system with N point masses, denoted by $m_k\,(k = 1, \ldots, N)$, where each mass does periodic orbital motion in the Newtonian gravitational field, whose gravitational constant is G. Assuming that each point mass has no angular rotation and collision with each other. The question becomes how to extract the basic parameters of the problem from these limited information. The success or failure of using dimensional analysis depends on how to select the basic parameters of the problem. If one choose the wrong parameters, it will lead to absurd results. Hence, for the n-body system, which parameters are the basic parameters?

In physics, these basic parameters should include the gravitational field, the characteristic area or length scale of the periodic orbit, as well as the characteristic mass of the system. It is clear that we can use the gravitational constant G to describe the Newtonian gravitational field; the characteristic mass can be the reduced mass μ_n, Each

cycle of the track is different, and not like the elliptical orbit of the semi-long axis, which is the characteristic scale, but the periodic orbit has a common topological feature, which is that all periodic orbit are closed, hence the area A_n of the closed orbit could be chosen as the characteristic area scale of the orbit, while its square root is the length scale. Now the periodic orbit problem of the n-body system then becomes to find the orbital period, T_n, and total energy, $|E_n|$, which is the summation of kinetic and potential energy, here takes its abstract value since it is negative for periodic orbit. From Newtonian gravitation theory, the attraction forces between bodies are linear proportional to the product of Gm_im_j, which means that the gravitation constant G can be absorbed into a new parameter α_n, whose dimension is $[L]^3[M][T]^{-2}$. In the following, we will use α instead of G as basic parameter.

The dimensions of those parameters are listed in the table below:

TABLE I: Parameters and Dimensions

α_n	μ_n	A_n	T_n	$\|E_n\|$
$[L]^3[M][T]^{-2}$	$[M]$	$[L]^2$	$[T]$	$[M][L]^2[T]^{-2}$

According to the dimensional analysis [13–17], the total energy $|E_n|$ can be expressed as

$$|E_n| = f(\alpha_n, \mu_n, A_n), \tag{1}$$

where f stands for a function. This relation has four parameters with three basic dimensions, namely time $[T]$, mass $[M]$ and length $[L]$. From Buckingham π-theorem [13–17], It can produce only one dimensionless parameter, $\pi = |E_n|\alpha_n^a\mu_n^bA_n^c$, the homogenous dimension theorem gives us $a = -1$, $b = 0$, $c = 1/2$, hence $\pi = |E_n|A_n^{1/2}/\alpha_n$, which must be a constant since it is one only. Therefore, we have

$$|E_n|A_n^{1/2}(\alpha_n)^{-1} = \text{const.} \tag{2}$$

In the similar way,the orbital period T_n can be expressed as $T_n = F(\alpha_n, M_n, A)$, where F stands for a function. This relation has four variables with three basic dimensions, namely time $[T]$, mass $[M]$ and length $[L]$. It can produce only one dimensionless parameter, $\pi = T_nA^{-3/4}\alpha_n^{1/2}\mu_n^{-1/2}$, which must also be a constant. Therefore n-body Kepler's third law is given by

$$T_nA_n^{-3/4}\alpha_n^{1/2}\mu_n^{-1/2} = \text{const.} \tag{3}$$

Combining the Eq.(2) and(3) and removing the area A_n, we can obtain a popular format of the Kepler's third law as follows

$$T_n|E_n|^{3/2} = \text{const.} \times \alpha_n\sqrt{\mu_n}. \tag{4}$$

It is noted that the period depends on the energy of the mass. The higher energy the system has, the shorter period it has. For each energy level, there is corresponding period, therefore infinite periodic orbits are existed [10, 11].

KEPLER'S THIRD LAW OF N-BODY SYSTEM

Although Eq.(4) has been formulated, we still cannot get much useful information if the constant, α_n and μ_n cannot be defined. In other words, the success and failure of Eq.(4) totally depends on determination of these three parameters. The situation is even more difficulty since the only analytic information is the 2-body Kepler's third law. Let's embark our journey from Kepler's third law.

Since dimensional result Eq.(4) is a general result and should also be true for 2-body system, we can determine the constant by this understanding. In Eq.(4), if we can set the reduced mass $\mu_2 = \frac{m_1m_2}{m_1+m_2}$ and parameter $\alpha_2 = Gm_1m_2$, by comparing with Kepler's third law $T|E|^{3/2} = \frac{\pi}{\sqrt{2}}G\left[\frac{(m_1m_2)^3}{m_1+m_2}\right]^{1/2}$, then we can propose that const.$=\frac{\pi}{\sqrt{2}}$.

If we carry on and will face a challenge problem, that is how to extend the 2-body Kepler's third law to n-body system? From Newtonian gravitation theory,for 2-body system (m_1, m_2), the attraction forces in between are proportional to the linear combination of m_1m_2; for 3-body system (m_1, m_2, m_3), the attraction forces between bodies are proportional to the linear combination of m_1m_2, m_1m_3 and m_2m_3; and for 4-body system (m_1, m_2, m_3, m_4), the attraction forces between bodies are proportional to the linear combination of m_1m_2, m_1m_3, m_1m_4, m_2m_3, m_2m_4, and m_3m_4; and for 5-body system $(m_1, m_2, m_3, m_4, m_5)$, the attraction forces between bodies are proportional to the linear combination of m_1m_2, m_1m_3, m_1m_4, m_1m_5, m_2m_3, m_2m_4, m_2m_5, m_3m_4, m_3m_5, and m_4m_5; and so on, for n-body, the attraction forces between bodies are proportional to the linear combination of m_im_j, $i = 1...n-1, j = i+1$. From mathematics of combination, the number of combination of mass product is $\binom{n}{2} = \frac{n!}{2(n-2)!}$.

We know that Kepler's third law

$$T_2|E_2|^{3/2} = \frac{\pi}{\sqrt{2}}G\left[\frac{(m_1m_2)^3}{m_1+m_2}\right]^{1/2}$$

, which has only one mass product m_1m_2 of 2-body system. However, for 3-body system we have three mass product combinations, namely

$$m_1m_2,\ m_1m_3,\ m_2m_3$$

, analogy to Kepler's third law, let's propose 3-body Kepler's third law as follows

$$T_3|E_3|^{3/2} = \frac{\pi}{\sqrt{2}}G\left[\frac{(m_1m_2)^3 + (m_1m_3)^3 + (m_2m_3)^3}{m_1+m_2+m_3}\right]^{1/2}. \tag{5}$$

Clearly, when $m_3 = 0$, Eq.(5) is reduced to 2-body Kepler's law. In physics, any n-body Kepler's law should be able to give 2-body Kepler's law, which means that the n-body Kepler's third law must be compatible with 2-body Kepler's law. In this regards, our formulation Eq.(5) is compatible with Kepler's third law.

In the light of Kepler's third law, we would like to propose following conjecture: n-body Kepler's third law could be expressed as follows

$$T_n|E_n|^{3/2} = \frac{\pi}{\sqrt{2}} G \left(\frac{S_n}{M_n}\right)^{1/2}. \tag{6}$$

where

$$S_n = \sum_{i=1}^{N} \sum_{j=i+1}^{N} (m_i m_j)^3$$

and total mass

$$M_n = \sum_{k=1}^{N} m_k$$

. Eq.(6) has answered the conjecture given by [6], namely, $T|E|^{3/2}$ = constant, the constant on the right-hand-side of this equation for a specific mass system is only a constant rather than "universal".

Some examples from Eq.(6) are listed in the following table III.

TABLE II: M_n and S_n of point mass system $(m_1, m_2, ..., m_n)$

N	M_n and S_n
2	$M_2 = m_1 + m_2$, $S_2 = (m_1 m_2)^3$
3	$M_3 = m_1 + m_2 + m_3$, $S_3 = (m_1 m_2)^3 + (m_1 m_3)^3 + (m_2 m_3)^3$
4	$M_4 = m_1 + m_2 + m_3 + m_4$, $S_4 = (m_1 m_2)^3 + (m_1 m_3)^3 + (m_1 m_4)^3 + (m_2 m_3)^3 + (m_2 m_4)^3 + (m_3 m_4)^3$
5	$M_5 = m_1 + m_2 + m_3 + m_4 + m_5$, $S_5 = (m_1 m_2)^3 + (m_1 m_3)^3 + (m_1 m_4)^3 + (m_1 m_5)^3 + (m_2 m_3)^3 + (m_2 m_4)^3 + (m_2 m_5)^3 + (m_3 m_4)^3 + (m_3 m_5)^3 + (m_4 m_5)^3$

NUMERICAL VALIDATION AND DISCUSSIONS

To compare with numerical simulation results carried out by [10, 11], numerical validation will be conducted for following cases.

Case 1: For 2-body system with $G = 1$ and $m_1 = m_2 = 1$, Kepler's third law gives $T_2|E_2|^{3/2} = \frac{\pi}{2} = 1.5707963$.

Case 2: For 2-body system with $G = 1$ and $m_1 = 1$ and m_2 varied. Kepler's third law gives $T_2|E_2|^{3/2} = \frac{\pi}{\sqrt{2}}(\frac{m_2^3}{1+m_2})^{1/2}$. This law is plotted in Fig 4 for different range of mass m_2. The law has obvious nonlinearity at $m_2 < 1$ and will be more linear as m_2 increasing.

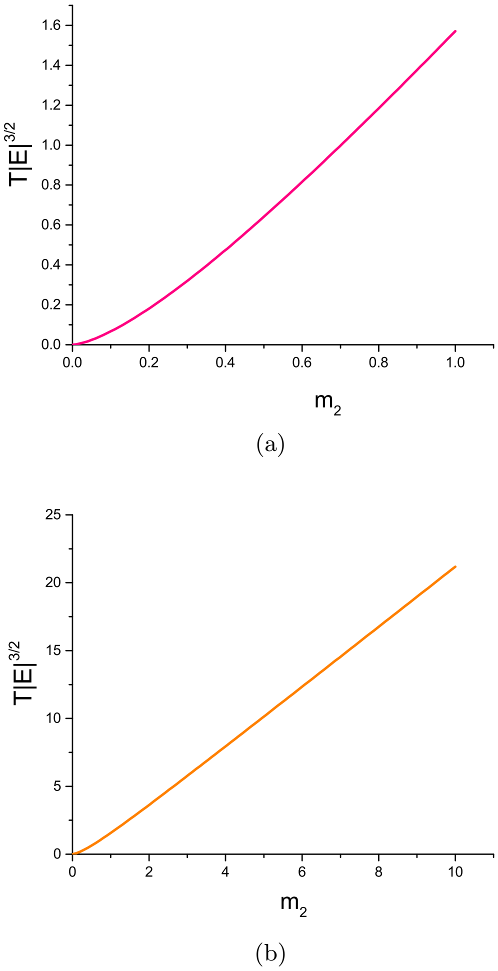


FIG. 4: Kepler's law for $G = 1$ and $m_1 = 1$ and m_2 varied

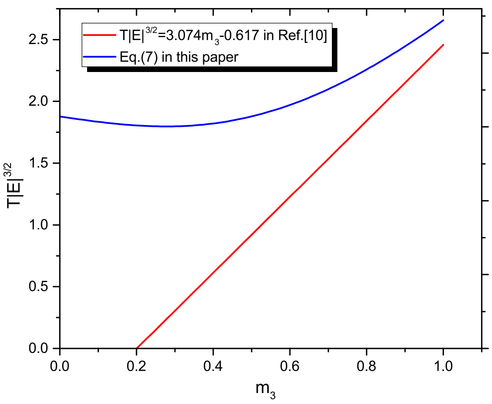
Case 3: For 3-body system with $G = 1$ and $m_1 = m_2 = m_3 = 1$, Li and Liao [10] obtained $T|E|^{3/2} = 2.433 \pm 0.075$ through numerical curve-fitting. In this case, Eq.(5) produces $T_3|E_3|^{3/2} = \frac{\pi}{\sqrt{2}} = 2.22144$. Our prediction is close to 2.358 with error 5.8%.

Case 4: For 3-body system with $G = 1$ and $m_1 = m_2 = 1$ and m_3 varied., Li, Jing and Liao's [11] proposed linear law $T|E|^{3/2} = 3.074m_3 - 0.617$. In this case, Eq.(5) gives

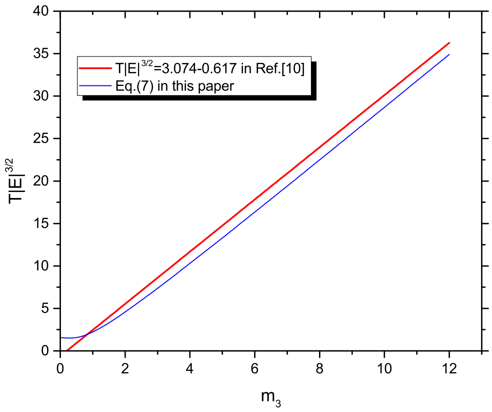
$$T_3|E_3|^{3/2} = \frac{\pi}{\sqrt{2}} \left[\frac{1 + 2(m_3)^3}{2 + m_3}\right]^{1/2}. \tag{7}$$

This relation indicates the Kepler's third law is nonlinear function of m_3. The numerical comparing are illustrated in Fig.5 and Fig.6.

The difference for $m_3 \in [0, 1]$ shown in Fig.(5), might be interpreted as follows: If set $m_3 = 0$, Eq.(7) gives 2-body Kepler's third law $T_2|E_2|^{3/2} = \frac{\pi}{2}$; however, the linear law $T|E|^{3/2} = 3.074m_3 - 0.617$ obtained by Li, Jing and Liao [11] gives $T|E|^{3/2} = -0.617$, which might have no physical meaning, since $T|E|^{3/2}$ should be positive.

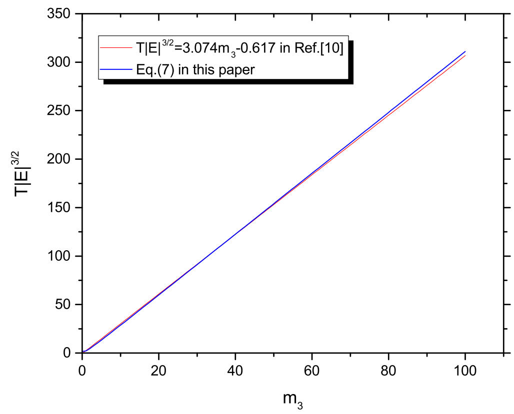


(a)

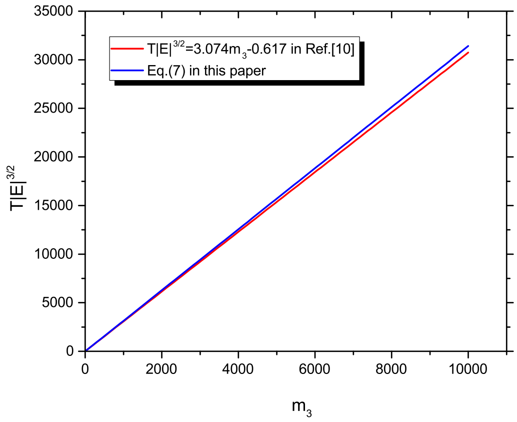


(b)

FIG. 5: Comparing for $m_3 \in [0, 12]$



(a)



(b)

FIG. 6: Comparing for $m_3 \in [10, 10000]$

Generally speaking, the figures show that our formulation has good linearity and keep same trends as the linear law $T|E|^{3/2} = 3.074m_3 - 0.617$ obtained by Li, Jing and Liao [11].

If we keep m_1 varied and set all other point masses to be unit mass, namely $m_k = 1$, $k \neq 1$, what's going to be happen?. Four cases are listed in the below table III.

TABLE III: M_n and S_n of point mass system $(m_1, 1, 1, 1, 1)$

N	m_k	M_n and S_n
2	m_1 varied $m_2 = 1$	$M_2 = m_1 + 1$, $S_2 = (m_1)^3$
3	m_1 varied $m_2 = m_3 = 1$	$M_3 = m_1 + 2$, $S_3 = 2(m_1)^3 + 1$
4	m_1 varied $m_k = 1$, $k = 2, 3, 4$	$M_4 = m_1 + 3$, $S_4 = 3(m_1)^3 + 3$
5	m_1 varied $m_k = 1$, $k = 2, 3, 4, 5$	$M_5 = m_1 + 4$, $S_5 = 4(m_1)^3 + 6$

Kepler's third law of those four cases have been illustrated in the Fig.(7), which indicate that the more point masses the system has the higher orbit period it has. In general, $T_{n+1}|E_{n+1}|^{3/2} > T_n|E_n|^{3/2}$ can be proven.

For a system with point masses $(M, 1, 1, 1, .., 1)$, if M were massive and much heavier than other unit mass,

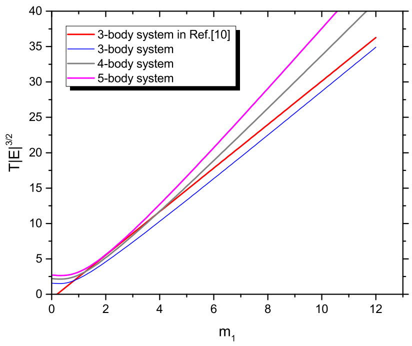


FIG. 7: $m_1 \in [0, 12]$

then we have interesting result as follows

$$T_n|E_n|^{3/2} \approx (\frac{n-1}{2})^{1/2}\pi GM. \tag{8}$$

Clearly, it is a linear law of single massive mass M.

CONCLUSIONS

In summary, this study considered the periodic orbital period of a n-body system from the perspective of dimensional analysis. The universal law of the n-body system is deduced: Kepler's third law, namely the periodic law, states that periodic motion of a n-body system satisfies the 3/4 power law of the orbital area, or the 3/2 power law of the total energy. In light of Kepler's third law, we proposed a compatible Kepler's third law for n-body system. A numerical validation and comparison study was hence conducted. This study may open a new venue for the investigation of multi-body system.

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[1] H. Poincaré, Acta Math. **13**, 5 (1890).
[2] Z.H. Xia, Annals of Mathematics, **135**(3)(1992).
[3] L.D. Landau and E. M. Lifshitz, Mechanics, (3rd ed.) (Butterworth-Heinemann, Oxford, 1976).
[4] C. Moore, Phys. Rev. Lett. **70**, 3675 (1993).
[5] A. Chenciner and R. Montgomery, Ann. of Math. **152** (2000) 881-901.
[6] M. Šuvakov and V. Dmitrašinović, Phys. Rev. Lett. **110**, 114301 (2013).
[7] V. Dmitrašinović, M. Šuvakov and A. Hudomal, Phys. Rev. Lett. **113**, 101102 (2014).
[8] V. Dmitrašinović and M. Šuvakov, American Journal of Physics. **82**, 609 (2014).
[9] M. Šuvakov and V. Dmitrašinović, Phys. Lett. A **379**, 1939-1945 (2015)
[10] X.M. Li and S.J. Liao, Sci. China-Phys. Mech. Astron. **60**, 129511 (2017).
[11] X.M. Li, Y.P Jing and S.J. Liao, arxiv.org/abs/1709.04775.
[12] L. Crane, Infamous 3-body problem has over a thousand new solutions, New Scientist magazine on 30 September 2017.
[13] P.W. Bridgman, *Dimensional Analysis* (Yale University Press, New Haven, 1922).
[14] L.I. Sedov, *Similarity and Dimensional Analysis in Mechanics* (Academic Press, New Yor, 1959).
[15] B. Sun, *Dimensional Analysis and Lie Group* (China High Education Press, Beijing, 2016).
[16] B. Sun, Phys. and Eng. (in Chinese) **26**, 6(2016).
[17] B. Sun, China-Phys. Mech. Astron. **60**, 104711(2017)
[18] T. Hecksher, Insights through dimensions, Nature Physics, **13**, 2017.
[19] E. Buckingham, Phil. Mag. **42**, 696 – 719 (1921).
[20] C. Marchal, J. Yoshida, and Y.S. Sun, **34**, 65 – 93(1984).